CENTRE OF MASS AND ROATIONAL MOTION

- Centre of mass of system of particles

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots . + m_n\vec{r}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

Here $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ etc are position vector of mass m_1 , m_2, m_3 .. etc

$\mathbf{r}_{CM}$ is position vector of centre of mass

- Velocity and momentum of centre of mass

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots . + m_n\vec{v}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots . + m_n\vec{v}_n}{M}$$

$$\vec{P} = M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots . + m_n\vec{v}_n$$

Here M is the total mass of system of particles

- Acceleration and force on centre of mass

$$\vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots . + m_n\vec{a}_n}{M}$$

$$\vec{F} = M\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots . + m_n\vec{a}_n$$

$$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_1} + \overrightarrow{F_1} + \cdots \overrightarrow{F_n}$$

- Circular motion

(i) Equation of motion θ : angular displacement, α : angular acceleration,

ω_0 : Initial Angular velocity; ω : Final Angular velocity, t: time

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega = \omega_0 + \alpha t$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$\theta = \left(\frac{\omega - \omega_0}{2}\right)t$$

$$\alpha = \frac{\omega - \omega_0}{t}$$

(ii) Angular momentum and Torque

$\overrightarrow{\boldsymbol{L}}$: Angular momentum ; τ ; torque

Angular momentum $\vec{\boldsymbol{L}} = \overrightarrow{\boldsymbol{r}} \times \vec{\boldsymbol{P}}$ = rpsinθ

Angular momentum $|\vec{\boldsymbol{L}}|$ = product of linear momentum and perpendicular distance between point of rotation and line of action.

Torque $\vec{\tau} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$

$$\tau = \frac{dL}{dt}$$

(iii) Moment of inertia and angular momentum

Moment of inertia I =

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2$$

Here r_1, r_2 etc are the perpendicular distance of particle from the axis of rotation

Angular momentum $\vec{\boldsymbol{L}}$ =I $\overrightarrow{\boldsymbol{\omega}}$

- Low of conservation of angular momentum

As $\overrightarrow{\boldsymbol{L}} = \overrightarrow{\boldsymbol{r}} \times \overrightarrow{\boldsymbol{P}}$ = rpsinθ

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt}$$

$$\frac{d\vec{L}}{dt} = I\vec{\alpha} = \vec{\tau}$$

If L is constant external torque is zero

(i) Its geometrical representation in planetary motion

if dA/dt is is an areal velocity then

$$\frac{dA}{dt} = \frac{L}{2m}$$

- Relation between linear and angular velocity

$\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \overrightarrow{\boldsymbol{r}}$

$$\frac{d\vec{v}}{dt} = \vec{r} \times \frac{d\vec{\omega}}{dt} + \frac{d\vec{r}}{dt} \times \vec{\omega}$$

$$\vec{a} = \vec{r} \times \vec{\alpha} + \vec{v} \times \vec{\omega}$$

Acceleration have two components

i) Tangential $a_T = r\alpha$

ii) Radial $a_r = v\omega$

iii) If angle θ = π/2

$$a = r\alpha + v\omega$$

$$a = r\sqrt{\alpha^2 + \omega^2}$$

- Equilibrium of a rigid body.

  For linear equilibrium $\sum F = 0$

  and for rotational equilibrium $\sum \tau = 0$

- (i)Theorem of perpendicular axis.

  $I_z = I_x + I_y$

  (ii) Theorem of Parallel axis.

  $I = I_{c.m} + Md^2$

  here M = total mass, d = distance between axis and axis passing through centre of mass

- Rolling down of body on an inclined plane. K = radius of gyration, R = radius

  (i) equations for velocity at the bottom of plane

$$v = \sqrt{\frac{2gh}{1 + \frac{K^2}{R^2}}}$$

  (ii) Equation for acceleration

$$a = \frac{g\ sin\theta}{1+\frac{K^2}{R^2}}$$

  (iii) Time taken to reach bottom

$$t = \sqrt{\frac{2s\left(1 + \frac{K^2}{R^2}\right)}{g\ sin\theta}}$$

  Condition for rolling without sliding

$$\mu_s \geq \frac{\tan\theta}{1 + \frac{K^2}{R^2}}$$

(i) For ring , (K = R)

$$\mu_s \geq \frac{1}{2} tan\theta$$

(ii) For disc , $K = \frac{R}{\sqrt{2}}$

$$\mu_s \geq \frac{1}{3} tan\theta$$

(iii)For solid sphere , $K = \frac{\sqrt{2}}{5} R$

$$\mu_s \geq \frac{2}{7} tan\theta$$

(iv) For rolling body of same mass and same radius

(Vel.)sphere > (Vel.)disc > (Vel.)shell ; (Vel.)ring

So acceleration

(v) Time taken to reach the bottom of the inclined plane must be just reverse

(t)sphere < (t)disc < (t)shell < (t)ring

- velocity, acceleration and time taken by some bodies to reach bottom, while rolling down , angle of inclination θ. S is the length of inclined plain

  (i) Body : Circular ring or cylindrical shell :

  Velocity at bottom $v = \sqrt{(gS \sin\theta)}$

  acceleration at bottom $a = \frac{1}{2} g \sin\theta$

  time taken to reach the bottom

$$t = \sqrt{\frac{4s}{gsin\theta}}$$

  (ii) Body : Circular disc or solid cylinder :

  Velocity at bottom $v = \sqrt{1.33\ g\ s\ sin\theta}$

  Acceleration at bottom $a = \frac{2}{3}\ g\ sin\theta$

Time taken to reach the bottom

$$t = \sqrt{\frac{3\ s}{g\ sin\theta}}$$

(iii) Body : Solid sphere

Velocity at bottom

$$v = \sqrt{\frac{10}{7}\ g\ s\ sin\theta}$$

Acceleration at bottom $a = \frac{5}{7}\ g\ sin\theta$

Time taken to reach the bottom

$$t = \sqrt{\frac{14\ s}{5g\ sin\theta}}$$

(iv) Body : Spherical shell

Velocity at bottom

$$v = \sqrt{\frac{6}{5}\ g\ s\ sin\theta}$$

Acceleration at bottom $a = \frac{3}{5}\ g\ sin\theta$

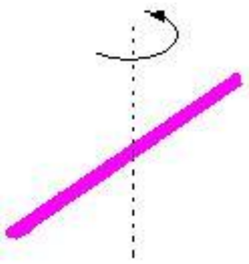
Time taken to reach the bottom

$$t = \sqrt{\frac{10\ s}{3g\ sin\theta}}$$

- Moment of inertia and radius of gyration for some symmetric bodies

(i) Body : Thin rod of length L

Diagram:Axis : Passing through its centre and perpendicular to length Moment



of inertia I :

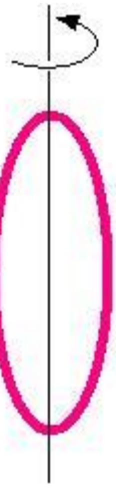
$$\frac{1}{12}ML^2$$

Radius of Gyration K :

$$\frac{L}{2\sqrt{3}}$$

(ii) Body : Ring of radius R

Diagram:



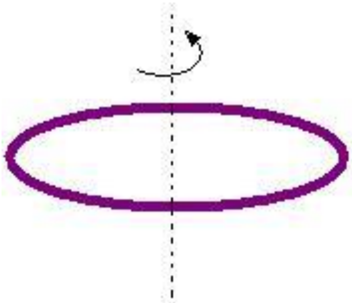
Axis : Any diameter

Moment of Inertia I : (1/2) MR^2

Radius of Gyration K : R / √2

(iii) Body : Ring of radius R

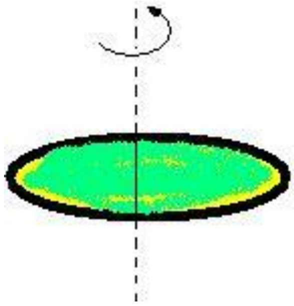
Diagram:



Axis : passing through its centre and perpendicular to its plane

Moment of Inertia I : MR^2

Radius of Gyration K : R



(iv) Body : Circular disc of radius R

Diagram:

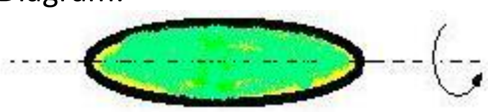
Axis : passing through its centre and perpendicular to its plane

Moment of Inertia I : (1/2) MR^2

Radius of Gyration K : R / √2

(v) Body : Circular disc of Radius R

Diagram:



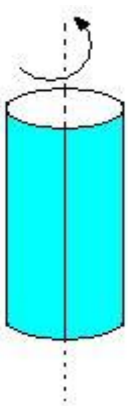
Axis: Any diameter

Moment of Inertia I : (1/4) MR^2

Radius of Gyration K : R/2

(vi) Body : Hollow cylinder of radius R

Diagram :



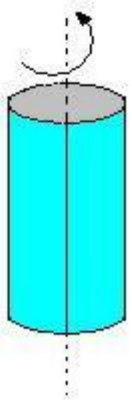
Axis : Geometrical axis of the cylinder

Moment of Inertia I : MR^2

Radius of Gyration K : R

(vii) Body : Solid cylinder of radius R

Diagram :

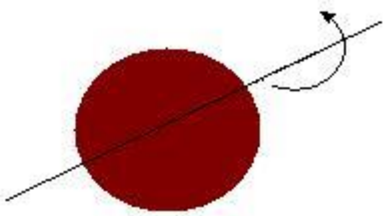


Axis : Geometrical axis of the cylinder

Moment of Inertia I : (1/2) MR^2

Radius of Gyration K : R / √2

(viii) Body :Solid sphere of radius R



Moment of inertia I :

$$\frac{2}{5}MR^2$$

Radius of gyration K:

$$\sqrt{\frac{2}{5}}R$$

(ix) Body: Hollow sphere of radius R

Axis : Any diameter

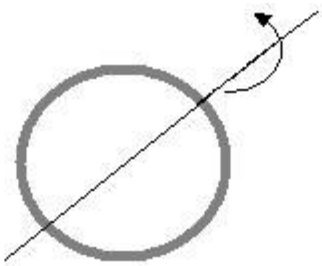


Diagram :

Moment of Inertia I :

$$\frac{2}{3}MR^2$$

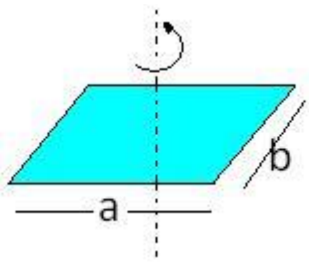
Radius of gyration K:

$$\sqrt{\frac{2}{3}}R$$

(x) Body : Rectangular lamina of length a and breadth b

Axis : passing through centre of gravity and perpendicular to plane

Diagram:



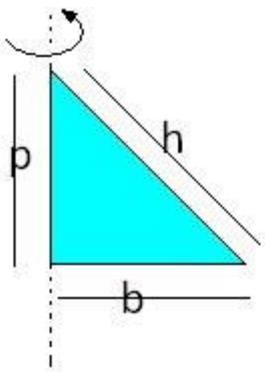
Moment of inertia I :

$$M\left(\frac{a^2+b^2}{12}\right)$$

Radius of gyration K:

$$\sqrt{\frac{a^2+b^2}{12}}$$

(xi) Body : Triangular lamina of mass M, base b, perpendicular p and hypotenuse h



Axis : About perpendicular

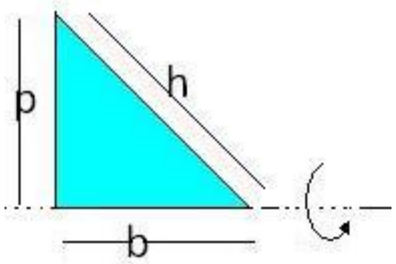
Moment of inertia I about axis p:

Radius of gyration K : $\frac{b}{\sqrt{6}}$

(xii) Body : Triangular lamina of mass M, base b, perpendicular p and hypotenuse h

Axis : About base

Diagram :



Moment of inertia I about axis b:

Radius of gyration K : $\frac{p}{\sqrt{6}}$

(xiii) Body : Triangular lamina of mass M, base b, perpendicular p and hypotenuse h

Axis : About hypotenuse

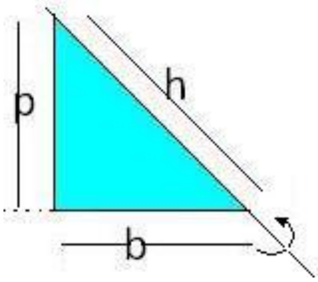


Diagram :

Moment of inertia I about hypotenuse :

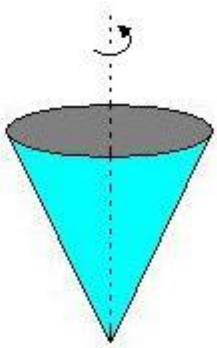
Radius of gyration K:

$$\frac{bp}{\sqrt{6\left(b^2-p^2\right)}}$$

(xiv) Body : Uniform cone of radius R and height h

Axis : through its C.G. and joining its vertex to centre of base

Diagram:



Moment of inertia I :

$$\frac{3}{10}MR^2$$

Radius of gyration :

- Kinetic energy of rolling body is E = K.E. translation (KT)+ K.E. rotational(KR)
  $$E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$
  (i) For Ring KT = KR

  (ii) For disc KT = 2KR

  (iii) For solid sphere KT = 5KR
- Graphical variation of parameters of rotatory motion

  (A) Parabolic

  (i) Rotational K.E and angular velocity(ω) as $KR \propto \omega^2$

(ii) Moment of inertia and radius of gyration as $I \propto K^2$

(iii) Anular momentum and Kinetic energy of rotation As $KR \propto L^2$

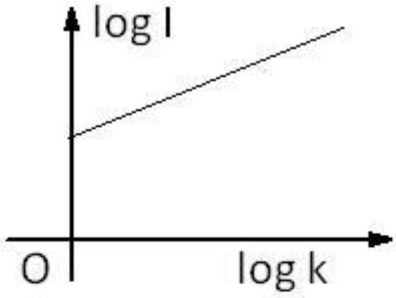
(B) Straight line

(i) Angular momentum (L) and angular speed (ω) As $L \propto \omega$

(ii) log I and log K

Moment of inertia $I = MK^2$

$\log I = \log M + 2\log K$



PROPERTIES OF MATTER

I) HEAT TRANSFER

- Heat transfer takes place in three ways i) Thermal conduction ii) Convection iii) Radiation

  (i)Thermal conduction is usually seen in solids

  (ii) Heat transfer takes place due to the difference in temperature between two adjacent parts .

  (iii) heat current H =

  $$H = \frac{\Delta Q}{\Delta t} = -kA\frac{\Delta T}{\Delta x}$$

  Here A is area of cross section, k is thermal conductivity and $\Delta T/\Delta x$ is called temperature gradient

  (iv) Thermal conductivity depends on the type of substance and temperature. Its Unit is $W\ m^{-1}\ K^{-1}$

- Thermal steady state

  (i) If in a substance, through which heat flows, the temperatures of each part are not constant (not same) the substance is said to be in thermal steady state

  At thermal steady state H = Q/t, if L is the length and $T_1 > T_2$ then

  $$H = \frac{kA(T_1 - T_2)}{L}$$

  (ii) Thermal Resistance RH = L / kA

  (iii) When two conducting rods are joined in series effective thermal resistance

  $$R_H = R_{H1} + R_{H2}$$

  (iv) When two conducting rods are joined in parallel effective thermal resistance

  $$\frac{1}{R_H} = \frac{1}{R_{H1}} + \frac{1}{R_{H2}}$$

- Absorptivity 'a' of surface : On irradiating a surface, the ratio of the radiant energy absorbed to the amount of radiant energy incident is called absorptivity 'a' of the surface.
- Emissivity : The ratio of total emissive power of surface to the total emissive power of the surface of perfectly black body under same condition is called emissivity 'e' of the surface
- Kirchhoff's law: The values of emissivity and absorptivity are equal for any surface i.e a = e .For black body a = e= 1
- Wien's displacement law: In blackbody radiation product of wavelength of a radiation having maximum spectral emissive power and absolute temperature is constant

  $\lambda_m T$ = constant

  Value of this constant = 2.9×10^{-3} mK

- Stefan Boltzmann's law : The amount of radiant energy emitted by surface per unit area per unit time is directly proportional to fourth power of its absolute temperature

  W = σeT4

  σ is Stefan-Boltzmann constant. Its value is $5.67 \times 10^{-8} Wm^{-2}K^4$
- Newton's law of cooling : the rate of loss of heat by a body depends on temperature difference ($T-T_s$) between body and its surrounding

$$\frac{dQ}{dt} \propto (T - T_s)$$

- Langmuir-Lorentz law : For natural convection the rate of cooling is proportional to $(5/4)^{th}$ power of difference of temperature between the body and its surroundings.

II) MECHANICAL PROPERTIES OF SOLIDs

Strain:

(i) longitudinal strain or tensile $\varepsilon_l = \Delta l/l$

(ii) Volume strain $\varepsilon_V = \Delta V/ V$

Shearing Strain $s = x/h = \tan\theta$

- Stress

  (i) Longitudinal stress $\sigma = F/A$

  (ii) Volume or Hydraulic stress $\sigma_V = F/A = PA/A = P$

  (iii) Shearing stress = Tangential force / area
- Hook's law : For small deformations the stress and strain are directly proportional to each other

  stress ∝ strain

  ∴ σ = kε

  k is known as modulus of elasticity . Its unit is Nm^{-2} or Pa

- Young's Modulus Y

$$Y = \frac{m\,g\,l}{\pi\,r^2\,\Delta l}$$

- Bulk Modulus B

$$B = -\frac{PV}{\Delta V}$$

Reciprocal of bulk modulus is compressibility (K)

- Modulus of rigidity(Shear Modulus) F_t is tangential force to surface

$$\eta = \frac{F_t\,h}{A\,x}$$

- Poisson's ratio : The ratio of lateral strain to longitudinal strain is known as Poisson's ratio denoted by μ

if lateral strain = Δd/ d and longitudinal strain = Δl/l

$$\frac{\Delta d}{d} = \mu \frac{\Delta l}{l}$$

$$\therefore \frac{\Delta r}{r} = -\mu \frac{\Delta l}{l}$$

- Change in volume due to longitudinal forces

$$\frac{\Delta V}{V} = \varepsilon_l \left(1 - 2\mu\right)$$

if ΔV > 0 value of μ can not exceed 0.5

- Elastic potential energy

Elastic potential energy = ½ (stress) × (strain) × Volume

- Bending of beam δ

W = Load, L : length , b= breadth and d = thickness

$$\delta = \frac{WL^3}{4bd^3Y}$$

- Relation between Y , B , η and σ

$i)\ Y = 3B\left(1-2\sigma\right)$

$ii)\ Y = 2\eta\left(1+\sigma\right)$

$iii)\ \sigma = \dfrac{3B-2\eta}{2\eta+6B}$

$iv)\ \dfrac{9}{Y} = \dfrac{1}{B} + \dfrac{3}{\eta}$

Thermal Stress: If two ends of rod are rigidly fixed and its temperature is changed, the length of the rod will change and so it will exert a normal stress on the supports.

Thermal stress = Y × thermal strain = $Y\alpha\Delta\theta$

α is coefficient of linear expansion

III) FLUID MECHANICS

- Units of pressure

(i) 1 Pa = $1Nm^{-2}$

(ii) 1 bar = 105 Pa

1 atm = 1.013 × 105 Pa

1 torr = 133.28 Pa

1 atm = 76cm Hg = 760 mm-Hg

- Pressure due to fluid column

$P = \rho gh$

If Pa denotes atmospheric pressure then (P-Pa) is known as gauge pressure or hydrostatic pressure at that point

- Buoyant force $F_b = \rho_f Vg$

Here ρ_f is density of fluid and V is the volume of body immersed or volume displaced by body

- Equation of Continuity $A_1v_1 = A_2 v_2$

  Here A_1 and A_2 are area of cross-section and v_1 and v_2 are velocity of fluid

  Bernoulli's equation

  $$\frac{P}{\rho g} + \frac{v^2}{2g} + h = cons.$$

  Here h is height ; v is velocity of fluid; P is pressure ; ρ is density of fluid

- Equation for Venturie meter :

  $$v = a\sqrt{\frac{2(\rho_l - \rho_m)gh}{\rho_l(A^2 - a^2)}}$$

  ρ_l : Density of fluid

  ρ_m : Density of Manometer fluid

  A : Cross -sectional area of pipe

  a ; cross-sectional area of throat

- Torricelli's law $v = \sqrt{2gh}$
- Viscous force

  $$F = \eta A \frac{dv}{dx}$$

  η : Coefficient of viscosity

  A : Area of contact

  dv/dx = velocity gradient

  CGS unit of coefficient of viscosity : dyne s cm^{-2}

  SI unit of coefficient of viscosity : N s m^{-2}

  or Pa s

  Dimensional formula of coefficient of viscosity : $M^1 L^{-1}T^{-1}$

  Viscosity depends on temperature and independent of pressure

- Stoke's Law : $F(v) = 6\pi\eta rv$

- Terminal velocity

  $$v_t = \frac{2}{9}\frac{r^2 g}{\eta}(\rho_s - \rho_l)$$

  ρ_s : density of body falling

  ρ_l : density of fluid

  r : radius of sphere

  η : coefficient of viscosity of fluid

- Reynold's Number

  $$N_R = \frac{\rho v D}{\eta}$$

  η : coefficient of viscosity of fluid

  D : Diameter of tube

  v : velocity of fluid

  If NR < 2000 : flow is streamline

  NR > 3000 : flow is turbulent

  for 2000 < NR < 3000 : flow is unstable

  Critical velocity : The maximum velocity for which the flow remains streamline is called critical velocity

- Poiseiulle's law :

  Volume of the liquid passing through the tube in one second

  $$V = \frac{\pi p r^4}{8\eta l}$$

  p : pressure difference across tube

  l : length of the tube

  r : radius of tube

  (a) $\frac{8\eta l}{\pi r^4} = R$

R called fluid resistance

(b) When two capillary tubes of different size are joined in series the equivalent fluid resistance is $R_s = R_1 + R_2$

(c) If two capillary tubes of different size are connected in parallel then equivalent fluid resistance is

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2}$$

- velocity of layer situated at 'x' from the axis of the tube

$$v = \frac{p}{4\eta l}\left(r^2 - x^2\right)$$

- Surface tension

(i) Work done to increase the surface W = SΔA

S: is surface tension , ΔA is increase in area

(ii) Pressure inside bubble :

$$P_i = P_o + \frac{4S}{R}$$

(iii) pressure inside drop

$$P_i = P_o + \frac{2S}{R}$$

(iv) Energy released while merge of n droplets each of radius r, to form a bigger drop of radius R, S is surface tension

$$W = 4\pi SR^3\left(\frac{1}{r} - \frac{1}{R}\right)$$

If two soap bubbles of radius r_1 and r_2 coalesce to form new bubble of radius r, under isothermal condition then

$$r = \sqrt{r_1^2 + r_2^2}$$

(v) height of liquid in capillary

$$h = \frac{2S\cos\theta}{r\rho g}$$

θ is angle of contact

(a) If θ < 90° , cosθ is positive, liquid rises up in the capillary example. glass -

water

(b) If $\theta > 90°$, $\cos\theta$ is negative, liquid falls in the capillary example. glass - mercury