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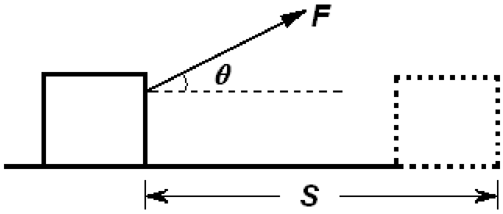
WORK, ENERGY AND POWER

Work done by a constant force:

Consider an object undergoes a displacement S along a straight line while acted on a force F that makes an angle θ with S as shown



The work done W by the agent is the product of the component of force in the direction of displacement and the magnitude of displacement

W = FScosθ

Work done is a scalar quantity and S.I. unit is N-m or Joule (J). Its dimensional formula is $M^1L^2T^{-2}$

We can also write; work done as a scalar product of force and displacement

$$W = \mathbf{F} \cdot \mathbf{S}$$

From this definition, we conclude the following points:

(i) Work done by a force is zero, if point of application of force does not move (S=0)

(ii) Work done by a force is zero if displacement is perpendicular to the force ($\theta=90^O$)

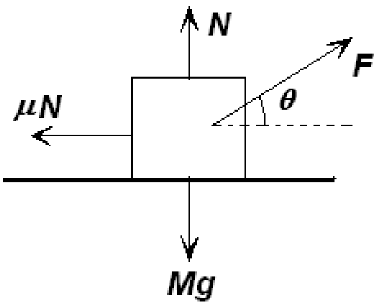
(iii) If angle between force and displacement is acute ($\theta <90^O$), we say that work done by the force is positive or work is done on the object

(iv) If angle between force and displacement is obtuse ($\theta > 90^O$), we say that work done by the force is negative or work is done by the object

Solved Numerical

Q) A block of mass M is pulled along a horizontal surface by applying a force at an angle θ with horizontal. Coefficient of friction between block and surface is μ. If the block travels with uniform velocity, find the work done by this applied force during a displacement d of the block

Solution



The forces acting on the block as shown in figure Force F will resolve as Fsinθ along normal while Fcosθ will be opposite to friction. Thus we get

Fcosθ = μN ----- (1)

And

N + Fsinθ = Mg -----eq(2)

Eliminating N from equation (1) and (2)

Fcosθ = μ (Mg - Fsinθ)

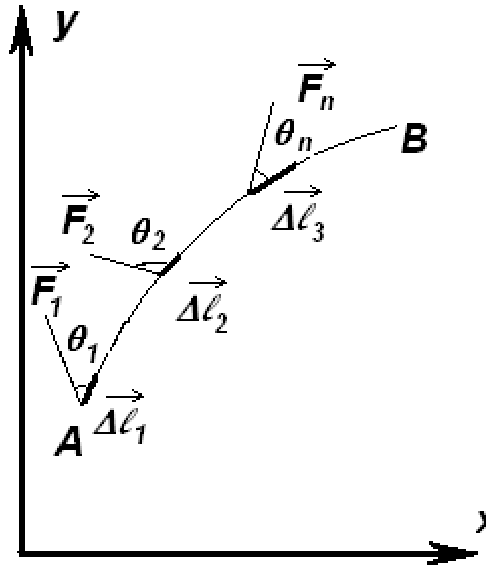
$$F\cos\theta + F\sin\theta = \mu mg$$

$$F = \frac{\mu Mg}{\cos\theta + \sin\theta}$$

Work done by this force during displacement d

$$W = Fd = \frac{\mu Mgd}{\cos\theta + \sin\theta}$$

Work done by Variable force



(a)

Consider a particle being displaced along the curved path under the action of a varying force, as shown in figure. In such situation, we cannot use W= (Fcosθ)S to calculate the work done by the force because this relationship applies when F is constant in magnitude and direction

However if we imagine that the particle undergoes a very small displacement $\Delta \mathbf{l_1}$, shown in figure(a), then F is approximately constant over this interval and we can express the work done by the force for this small displacement as $W_1 = F_x \Delta \mathbf{l_1}$

In order to calculate work done, the whole curved path is assumed to be divided in small segments $\Delta \mathbf{l_1}$, $\Delta \mathbf{l_2}$, $\Delta \mathbf{l_3}$, ---------$\Delta \mathbf{l_n}$

Let $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, --------- $\mathbf{F}_n$ be the force at respective segments. The force over each such segment can be considered as constant because the segments are very small.

Total work done

$W = \mathbf{F}_1 \cdot \Delta \mathbf{l_1} + \mathbf{F}_2 \cdot \Delta \mathbf{l_2} + \mathbf{F}_3 \cdot \Delta \mathbf{l_3} + \ldots\ldots\ldots\ldots + \mathbf{F}_n \cdot \Delta \mathbf{l_n}$

$$W = \sum_{A}^{B} \vec{F}_i \overrightarrow{\Delta l}_i$$

If we take $|\Delta \mathbf{l}| \rightarrow 0$, the above summation gets converted into an integral

$$W = \int_{A}^{B} \vec{F} \cdot \overrightarrow{dl} = \int_{A}^{B} F\cos\theta dl$$

Solved Numerical

Q) A particle moves from x = 0 to x = 10m on X-axis under the effect of force

$F(x) = (3x^2 - 2x + 7)\mathbf{i}$ N.

Calculate the work done

Solution: since direction of force and displacement is same θ = 0

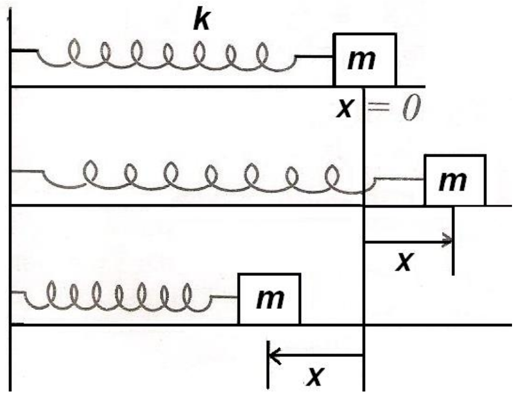
$$W = \int_{0}^{10} F\, dx$$

$$W = \int_{0}^{10} (3x^2 - 2x + 7)\, dx$$

$$W = \left[\frac{3x^3}{3}\right]_0^{10} - \left[\frac{2x^2}{2}\right]_0^{10} + [7x]_0^{10}$$

W = 1000 - 100 + 70 = 970 J

Work done by a spring



A common physical system for which the force varies with position is a spring-block as shown in figure. If the spring is stretched or compressed by a small distance from its unstitched or compressed by a small distance from its unscratched configuration, the spring will exert a force on the block given by F = -kx, where x is compression or elongation in spring, k is a constant called spring constant whose value depends inversely on un-stretched length and the nature of material of spring.

Negative sign in above equation indicates that the direction of the spring force is opposite to x , the displacement of the free end.

Consider a spring block system as shown in figure and let us calculate work done by the spring when block is displaced by x_0

At any moment if elongation is x, then force on block by spring is kx towards left.

Therefore, work done by the spring when block further displaced by dx

dw = - kxdx (Negative sign indicates displacement is opposite to spring force)

Total work done by the spring

$$W = -\int_{0}^{x_0} kx\, dx = -\frac{1}{2}kx_0^2$$

Similarly, work done by the spring when it is given a compression x_0 is

$$-\frac{1}{2}kx_0^2$$

We can also say that work done by external agent

$$\frac{1}{2}kx_0^2$$

Power

If external force is applied to an point like object and if the work done by this force is ΔW in the time interval Δt, then the average power during this interval is defined as

$$P = \frac{\Delta W}{\Delta t}$$

The work done on the object contributes to increasing energy of the object. A more general definition of power is the time rate of energy transfer. This instantaneous power is the limiting value of the average power as Δt approaches zero

$$P = \frac{dW}{dt}$$

Where we have represented the infinitesimal value of the work done by dW (even though it is not a change and therefore not a differential). We know that

$dW = \vec{F} \cdot \vec{S}$

Therefore the instantaneous power can be written as

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{S}}{dt} = \vec{F} \cdot \vec{v}$$

The SI unit of power is Joule per second (J/s), also called watt (W)

1W = 1 J/s = 1kgm^{2}s^{-3}

Energy

A body is said to possess energy if it has the capacity to do work. When a body possessing energy does some work, part of its energy is used up.

Conversely if some work is done upon an object, the object will be given some energy.

Energy and work are mutually convertible.

Kinetic energy

Kinetic energy (K.E.) is the capacity of a body to do work by virtue of its motion

If a body of mass m has velocity v its kinetic energy is equivalent to the work, which an external force would have to do to bring the body from rest to its velocity v.

The numerical value of the kinetic energy can be calculated from the formula

$$K.E. = \frac{1}{2}mv^2$$

This formula can be derived as follows:

Consider a **constant force F** which acting on a mass m initially at rest, particle accelerate with constant velocity and attend velocity v after displacement of S .

For the formula

$v^2 - u^2 = 2as$

Initial velocity is zero

$v^2 = 2as$

Multiply both the sides by m

$mv^2 = 2mas$

$mv^2 = 2W$ [As work = FS = mas]

$W = (1/2)mv^2$

But Kinetic energy of body is equivalent to the work done in giving the velocity to the body

Hence K.E = $(1/2)mv^2$

Since both m and v^2 are always positive K.E is always positive and does not depend up on the direction of motion of body. Another equation for kinetic energy

$$E = \frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2}{m} = \frac{1}{2}\frac{p^2}{m}$$

Potential energy

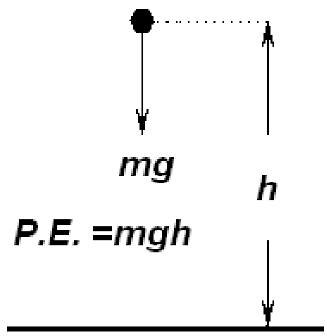
Potential energy is the energy due to position. If a body is in a position such that if it were released it would begin to move, it has potential energy

There are two common forms of potential energy, gravitational and elastic

Gravitational potential energy

It is possessed by virtue of height

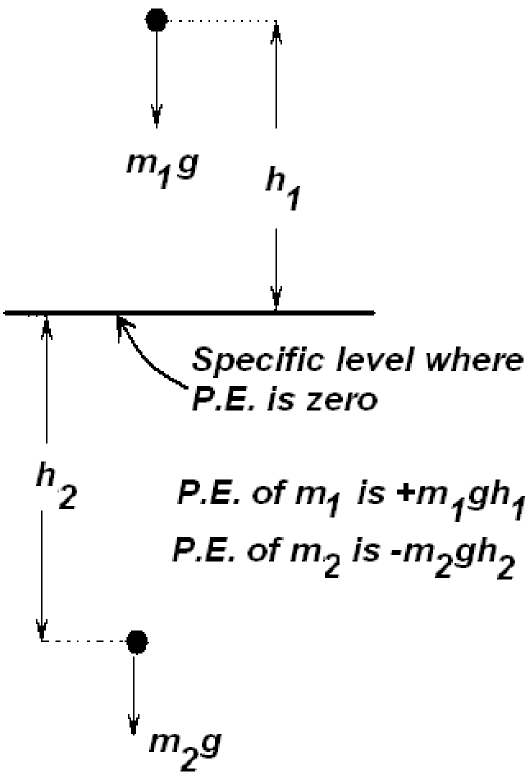
When an object is allowed to fall from one level to a lower level it gains speed due to gravitational pull, i.e. it gains kinetic energy. Therefore, in possessing height, a body has the ability to convert its gravitational potential energy into kinetic energy.



The gravitational potential energy is equivalent to the negative of the amount of work done by the weight of the body in causing the descent.

If a mass m is at a height h above a lower level, the P.E. possessed by the mass is (mg) (h)

Since h is the height of an object above a specific level, an object below the specified level has negative potential energy



Therefore Gravitational Potential Energy = $\pm$ mgh

- The chosen level from which height is measured has no absolute position. It is important therefore to indicate clearly the zero P.E. level in any problem in which P.E. is to be calculated.
- Gravitational Potential Energy = $\pm$ mgh is applicable only when h is very small in comparison to the radius of earth.

Elastic potential Energy

It is a property of stretched or compressed springs.
The end of a stretched elastic spring will begin to move if it is released. The spring therefore possesses potential energy due to its elasticity (i.e. due to change in its configuration)
The amount of elastic potential energy stored in a spring of natural length a and spring constant k when it is extended by a length x is equal to the amount of work necessary to produce the extension
Work done = $(1/2)kx^2$ so
Elastic Potential energy = $(1/2)\ kx^2$
Elastic potential energy is never negative whether the spring is extended or compressed

Work energy theorem

When a body is acted upon by force acceleration is produced in it. Thus velocity of the body changes and hence the kinetic energy of the body also changes. Also force acting on a body displaces the body and so work is said to be done on the body by force. These facts indicate that there should be some relation between the work done on body and change in its kinetic energy.
The work done by the force F
W= F S
W = ma
W = mas
Also $v^2 - u^2 = 2as$
Multiplying both sides by m
$m(v^2 - u^2) = 2ams$

$$\frac{1}{2}mv^2 - \frac{1}{2}u^2 = mas$$

$$\frac{1}{2}mv^2 - \frac{1}{2}u^2 = W$$

Here u and v are the speeds before and after application of force.
The left hand side of above equation gives change in kinetic energy while right hand gives the work done
Thus $\Delta K = W$
The work done by the resultant force on a body is equal to change in kinetic energy of the body. This statement is known as work energy theorem.

Work energy theorem for variable force
Work-energy theorem is valid from variable force
Suppose position dependent force F(x) acts on a body of mass m
Work done under the influence of force

$$W = \int_i^f F(x)dx$$

$$W = \int_i^f m\frac{dv}{dt}dx$$

$$W = \int_i^f m\frac{dx}{dt}dv$$

$$W = \int_i^f mv\,dv \quad [\because \frac{dx}{dt} = v]$$

$$W = m\int_i^f v\,dv$$

If initial velocity of the body and final velocity of the body are v_i and v_f

$$W = m\int_{v_i}^{v_f} v\,dv = m\left[\frac{v^2}{2}\right]_{v_i}^{v_f}$$

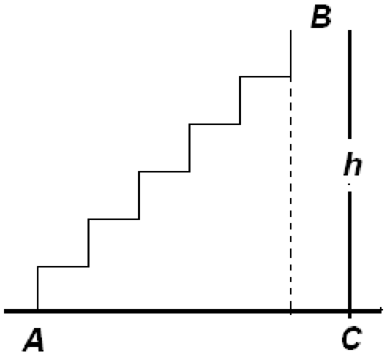
$$W = \frac{m}{2}\left[v_f^2 - v_i^2\right]$$

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Conservative and non-conservative force

Conservative force

A conservative force may be defined as on for which work done in moving between two points A and B is independent of the path taken between two points. Work done to move particles through stairs is equal to moving particle vertically .The implication of "conservative" in this context is that you could move it from A and B by one path and returns to A by another path with no net loss of energy – any close return path A takes net work zero. Or mechanical energy is conserved



A further implication is that the energy of an object which is subject only to that conservative force is dependent upon its position and not upon the path by which it reached that position. This makes it possible to define a potential energy function which depends upon position only

If a force acting on an object is a function of position only, it is said to be a conservative force and it can be represented by potential energy function which for a one-dimensional case satisfies the derivative condition

$$F(x) = -\frac{dU}{dx}$$

Example for verification

(a) Gravitational potential energy = -mgh

Thus

$$F(h) = -\frac{d(-mgh)}{dh}$$

F(h) = mg

(b) Spring potential energy = (1/2)kx^2

$$F(x) = -\frac{1}{2}k\frac{d(x^2)}{dx}$$

$$F(x) = -kx$$

Non-conservative force

Consider a body moving on a rough surface from A to B and then back from B to A. Work done against frictional forces only add up because in both the displacement work is done against frictional force only. Hence frictional force cannot be considered as a conservative force. It is non-conservative force

Conservation of mechanical energy

Kinetic and potential energy both are forms of mechanical energy. The total mechanical energy of a body or system of bodies will be changed in values if

(a) An external force other than weight causes work to be done(work done by weight is potential energy and is therefore already included in the total mechanical energy)

(b) Some mechanical energy is converted into another form of energy (e.g. sound, heat , light) such a conversion of energy usually takes place when a sudden change in the motion of the system occurs. For instance, when two moving objects collide some mechanical energy is converted into sound energy, which is heard as a bang at the impact.

If neither (a) nor (b) occurs, then the total mechanical energy of a system remains constant.

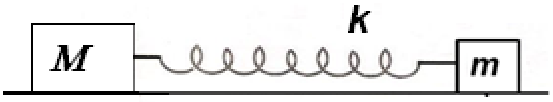
This is the principle of Conservation of Mechanical Energy and can be expressed as

The total mechanical energy of a system remains constant provided that no external work is done and no mechanical energy is converted into another form of energy

When this principle is used in solving problems, a careful appraisal must be made of any external forces, which are acting. Some external forces do work and hence cause a change in the total energy of the system.

Solved Numerical

Q) A spring of force constant k is kept in compressed condition between two blocks of masses m and M on the smooth surface of table as shown in figure. When the spring is released both the blocks move in opposite directions. When the spring attains its original (normal) position, both the blocks lose the contacts with spring. If x is the initial compression of the spring find the speed of block while getting detached from the spring.



Solution

According to law of conservation of energy

Spring Potential energy = Sum of kinetic energy off block

$$\frac{1}{2}kx^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2$$

From law of conservation of momentum

$mv_1 = Mv_2$

$$v_2 = \frac{m}{M}v_1$$

$$kx^2 = mv_1^2 + M\left(\frac{m}{M}v_1\right)^2$$

$$kx^2 = v_1^2\left(m + \frac{m^2}{M}\right)$$

$$kx^2 = v_1^2\left(\frac{mM + m^2}{M}\right)$$

$$v_1^2 = \frac{kx^2M}{m(M+m)}$$

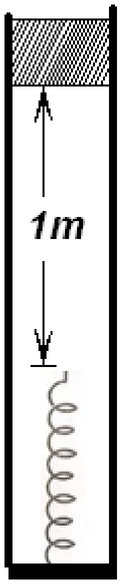
$$v_1 = \sqrt{\frac{kM}{m(M+m)}}\cdot x$$

Similarly

$$v_2 = \sqrt{\frac{km}{M(M+m)}}\cdot x$$

Q) A 20kg body is released from rest, so as to slide in between vertical rails and compresses a spring having a force constant k = 1920 N/m. the spring is 1m below the starting position of the body. The rail offers a resistance of 36N to the motion of the body. Find (i) the velocity of the body just before touching the spring (ii) the distance ,ℓ through which the spring is compressed (iii) the distance ‘h’ through which the body rebounds up

Solution



(i) Let velocity of the body just before touching the spring be v

Change in k.E = work done

$$\frac{1}{2}mv^2 - 0 = mg \times 1 - 36 \times 1$$

$$\frac{1}{2} \times 20 \times v^2 = 20 \times 9.8 \times 1 - 36 \times 1$$

v = 4 m/s

(ii) Let x be maximum compression of the spring. Then effective height for calculation of potential energy = 1+x

From conservation of energy

Spring potential energy = Change in P.E - Work done against friction

$$\frac{1}{2}kx^2 = mg(1+x) - 36(1+x)$$

$$\frac{1}{2} \times 1920 \times x^2 = 20 \times 9.8 \times (1+x) - 36(1+x)$$

X = 0.5 m

(iii) Let object bounce up to height h

Potential energy of object = spring potential energy – work against friction

$$mgh = \frac{1}{2}kx^2 - 36h$$

$$20 \times 9.8 \times h = \frac{1}{2} \times 1920 \times (0.5)^2 - 36h$$

$$h = \frac{240}{232} = \frac{30}{29} = 1.03m$$

Q) if work is done on a particle at constant rate, prove that the velocity acquired in describing a distance from rest varies as $x^{1/3}$

Solution

Power is constant, P = F .V = constant (say k)

Now mav = K

$$av = \frac{k}{m}$$

$$v\frac{dv}{dt} = \frac{k}{m}$$

$$v\frac{dv}{dt}\frac{dx}{dx} = \frac{k}{m}$$

$$v^2\frac{dv}{dx} = \frac{k}{m}$$

$$v^2 dv = \frac{k}{m}dx$$

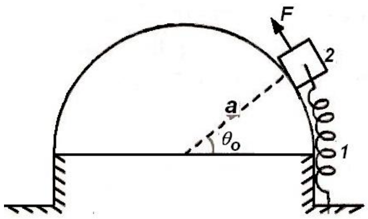
$$\int_0^v v^2 dv = \frac{k}{m}\int_0^x dx$$

$$\frac{v^3}{3} = \frac{k}{m}x$$

$$v^3 \propto x$$

$$v \propto x^{1/3}$$

Q) In the position shown in figure, the spring constant k is undeformed. Find the work done by the variable force F, which is always directed along the tangent to the smooth hemispherical surface on the small block of mass m to shift it from the position 1 to position 2 slowly.



Solution:

From the condition of the equilibrium of the block at any arbitrary angular position $\theta < \theta_o$

F = mgcosθ + kx

Work done in displacing the block through a distance dx = dW

dW = Fdx = (mgcosθ + kx)dx

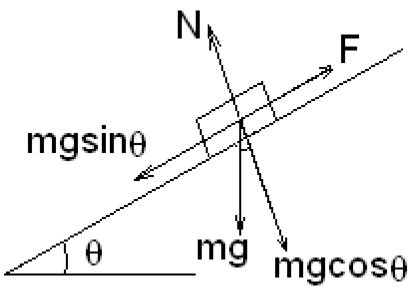
where x = aθ, dx = adθ

Total work done by the force F on the small block of mass m to shift from the position 1 to position 2 is

$$W = \int F dx = \int_{\theta}^{\theta_0} (\mathrm{mgcos\theta\ +\ ka\theta)ad\theta}$$

$$W = mgasin\theta_0 + \frac{ka^2}{2}\theta_0^2$$

Q) A block of mass 2 kg is pulled up on a smooth incline of angle 30^O with horizontal. If the block moves with an acceleration of 1 m/s^2, find the power delivered by the pulling force at a time 4 seconds after motion starts. What is the average power delivered during these four seconds after the motion starts?



Solution:

To find power delivered by force at t=4 we have to calculate velocity at t=4 and use formula P = force × velocity

i) Calculation of velocity at t = 4 s

$$V = u + at$$

$$a = 1 m/s^2 \ t = 4 \text{ sec given}$$

$$V = 4 m/s$$

ii) Calculation of resultant force

Given resultant acceleration $a = 1 m/s^2, \theta = 30^O$, $g = 9.8 m/s^2$

Thus from the diagram and resolving forces we get equation

$F - mg\sin\theta = ma$

$F = mg\sin\theta + ma$

$F = 2\times9.8\times\sin30 + 2\times1 = 11.8 N$

By substituting values of F and v in equation of power

P = Fv

P = 11.8 × 4 = 47.2 W

To find average power

We have to find total work done by using formula W = FS and then use formula P = W/t

But S is not given , can be calculated using $v^2 = u^2 + 2as$

We have already calculated v = 4m/s, u = 0, a = 1m/s^2

Thus

$$16 = 0 + 2\times1\times S$$

$$S = 8 \text{ m}$$

Now work done in 4 seconds = Force × displacement

Work done in 4 seconds = 11.8 ×8 = 94.4 J

Average power = Work / time

Average power = 94.4 / 4 = 23.6 W

Q) A block of mass m released from rest onto an ideal non-deformed spring of spring constant k from a negligible height. Neglect the air resistance, find the compression d of the spring.

Solution

(Note: When we attach or put mass on spring, spring under goes motion hence can not solve using formula mg = kx which is the condition for equilibrium)

Block is just kept on spring not allow to fall on spring. Thus Weight of block will press the spring and restoring force will oppose the compression. And equilibrium will be establish.

Let compression be 'd' thus potential energy lost by the block = mgd

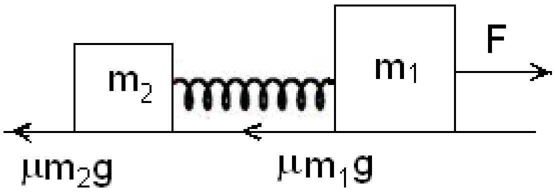
Potential energy gain by spring = $(/1/2)kd^2$

Thus potential energy lost by the block = potential energy gain by spring

$$mgd = (1/2)kd^2$$

$$d = 2mg/k$$

Q) Two masses m_1 and m_2 connected by a non-deformed light spring rest on a horizontal plane. The coefficient of friction between bars and surface is μ. What minimum constant force has to be applied in the horizontal direction to the mass m_1 in order to shift the other mass m_2



Solution:
Note that acceleration of both the masses will be different. Because acceleration of mass m_2 is due to restoring force of spring.
Problem can be solved using law of conservation of energy.
Fist consider mass m_2 do not move and is stationary.
Let x be the displacement of mass m_1 then Work done by force = Fx
This work done is used to overcome friction of mass m_1 and remaining stored as potential energy of spring
Now work done to overcome friction = Frictional force × displacement

$$= \mu\, m_1 g\, x$$

Energy stored in spring since mass m_1 have moved by distance 'x', stretching is spring is 'x' as we have already stated motion of block m_2 is due to restoring force of spring
Thus PE. Of Spring = $(1/2)kx^2$

$$Fx = \mu\, m_1 g\, x + (1/2)kx^2 \quad \text{-- --- eq(1)}$$

Since block m_2 moves due to restoring force of spring thus restoring force = frictional force

$$Kx = \mu\, m_2 g.$$

Substituting value of Kx in equation (1) we get

$$Fx = \mu\, m_1 g\, x + (1/2)\, \mu\, m_2 g x$$

$$F = \mu\, g\, (\, m_1 + m_2 /2\,)$$

Q) A block of mass M is attached with a vertical relaxed spring of spring constant k. if the block is released, find maximum elongation in spring.
Solution: Let x be the elongation.
Thus potential energy lost by mass = mgx
Energy gain by spring = $(1/2)\, kx^2$
From law of conservation of energy
Potential energy loss by mass M = energy gain by spring
$Mgx = (1/2)kx^2$
$X = 2Mg/k$
(Note: When we attach or put mass on spring, spring under goes motion hence can not solve using formula mg = kx which is the condition for equilibrium)

Q) A horse pulls a wagon of 5000 kg from rest against a constant resistance of 90N. the pull exerted initially is 600N and it decreases uniformly with the distance covered to 400N at a distance of 15m from start. Find the velocity of wagon at this point.
Solution:
Force is varying Initially 600 N and goes down to 400 N. Thus average force applied for pull = (600 +400) /2 = 500 N
Resistive force is constant 90N
Effective force = Average force – resistive force
Effective force = 500 – 90 = 410 N
Displacement = 15 m
Thus work done by the force = 410 × 15 = 6150 J
This work by force produces kinetic energy
∴ Kinetic energy of object = work done by force

$$\therefore \ \frac{1}{2}mv^2 = 6150$$

V= 1.57 m/s

Q) A block of mass 5.0 kg is suspended from the end of a vertical spring, which is stretched by 10cm under the lead of the block. The block is given is given a sharp impulse from below so that it acquires an upward speed of 2.0 m/s. How high will it rise? Take g = 10 m/s^2
Solution:
For equation mg = kx
5 ×10= k ×0.1
K = 500 N
Spring is already elongated by 0.1m thus it already have some potential energy, block attached to spring also have potential energy.
when sharp impulse is given to block gain more potential energy and spring gain potential energy
Thus After sharp impulse given Total energy of system
= previously store potential energy of spring due to 0.1m elongation + given kinetic energy+ potential energy of mass
[Here potential energy of mass is taken positive as below equilibrium point of spring before attaching mass]
When spring gets compressed say by x
= Potential energy of spring + potential energy of block
= $(1/2)kx^2$ - mg(x)
[Here potential energy taken negative as object moved above the equilibrium position of spring before attaching mass]

Thus from law of conservation of energy
$(1/2)\ k(0.1)^2 + (1/2)\ mv^2 + mg(0.1) = (1/2)kx^2 - mg(x)$
$(1/2)500(0.01) + (1/2)5(2)^2 + 5(10)(0.1) = (1/2)(500)x^2 – 5(10)x$
X = 0.1 m and height to which block raise = 0.1 +0.1= 0.2m from equilibrium point before attaching the mass.